METR 5223: Atmospheric Radiation

The radiative-transfer equation

Lecture for Spring 2009

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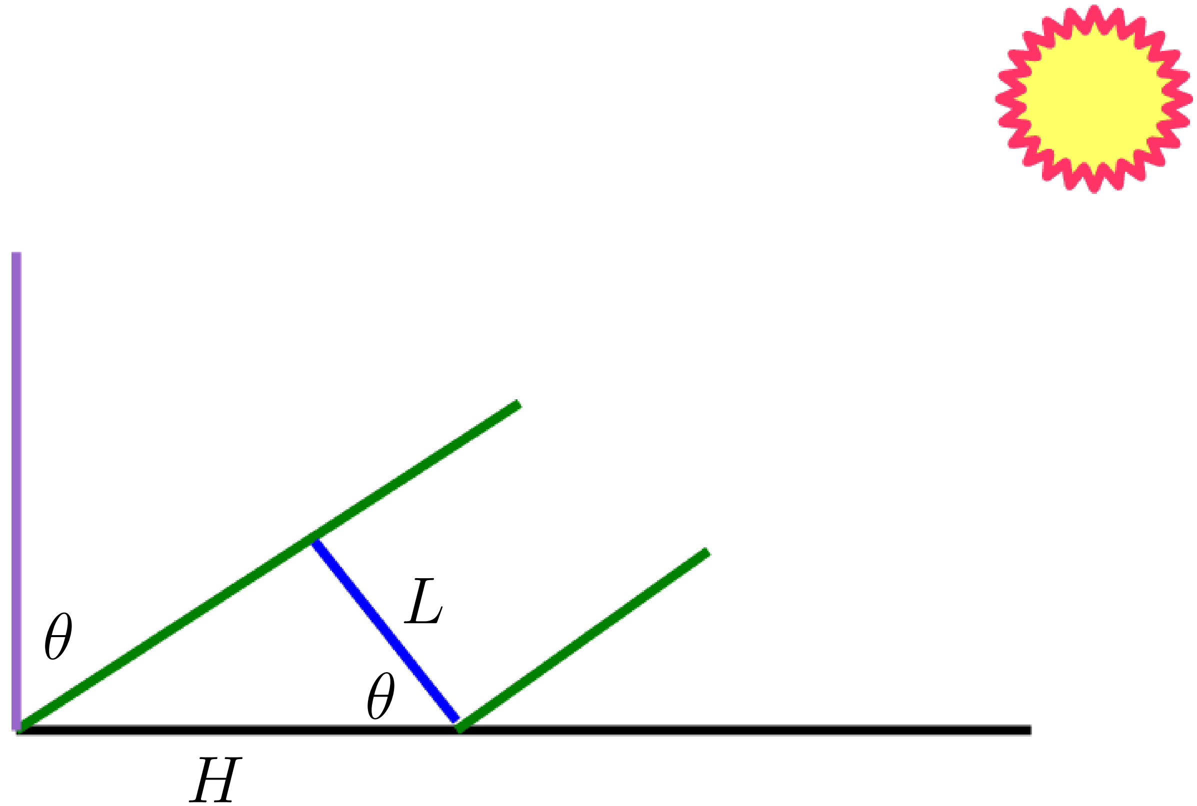
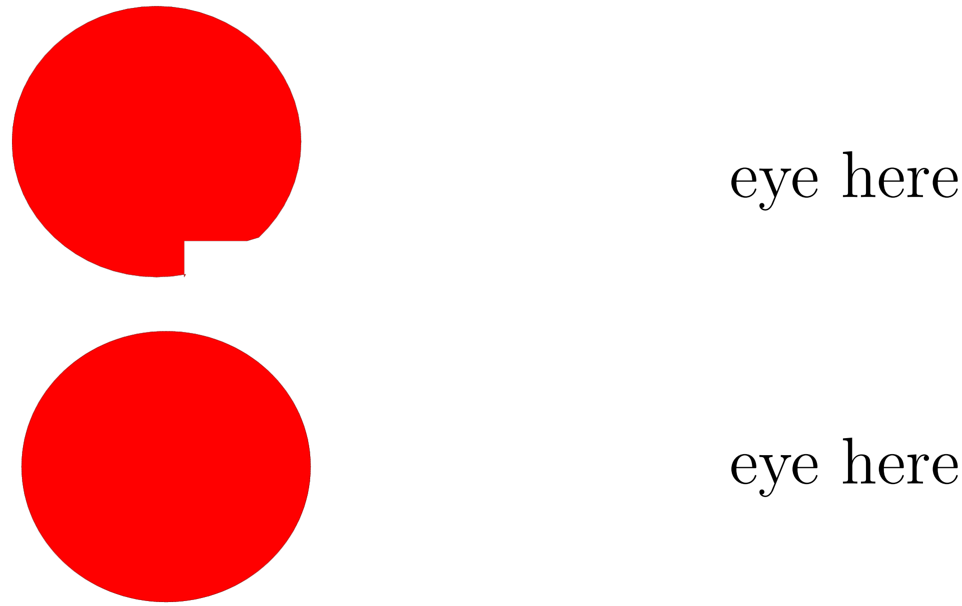
The **zenith angle** θ



Conservation of Energy: $LS_0 = HF$

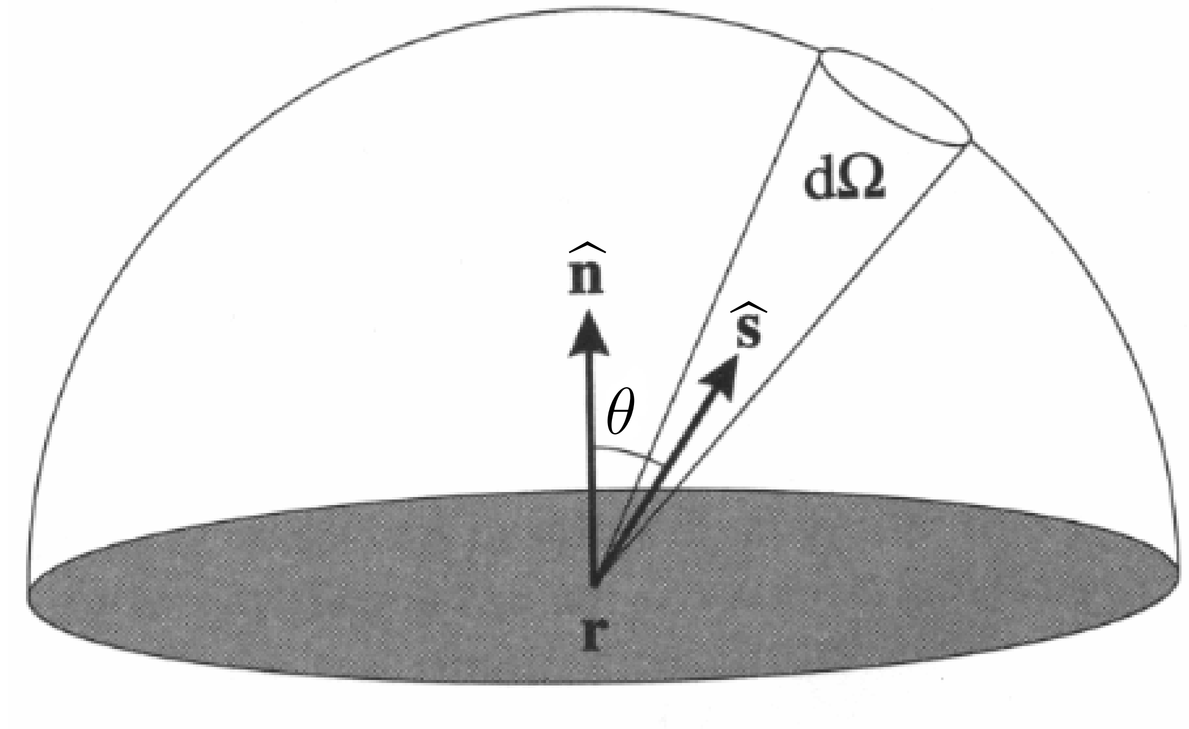
Using $L = H\cos\theta$,

$$F = S_0 \cos\theta$$



The brightness of the Sun is almost the same everywhere on an image of the solar disk. Evidently, emission rate is proportional to cosine of angle between normal of the surface and the eye. Thus, if the sun had a notch, with less surface area visible to you, being Lambertian it would have same color and intensity.

The Moon is NOT a Lambertian reflector. The incident radiation is reduced by a factor of cosine of the zenith angle toward the limb. Yet when the sun is behind us, the brightness is nearly the same across the disk!



The fundamental definition, **spectral radiance**, $L_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}})$ in W m^{-2} Hz^{-1} sr^{-1} (W&H: $L \rightarrow I$).

To understand spectral radiance, ignore the gray disc and its normal $\widehat{\mathbf{n}}$ in the above figure for now. $\vec{\boldsymbol{r}}$ is the position vector; all points in space have an L_ν. At $\vec{\boldsymbol{r}}$, L_ν will depend on direction $\widehat{\mathbf{s}}$...

Consider the cone having a certain finite area at its apex, with a normal $\widehat{\mathbf{s}}$ down the cone. The power moving down the cones decreases as this area decreases. But the limit of power per area (W m^{-2}) reaches a limit as the area at the apex goes to zero. Similarly, as the solid angle of the cone is decreased, restricting the radiation to be closer in direction to $\widehat{\mathbf{s}}$, the power decreases. But the power per solid angle converges to a limit.

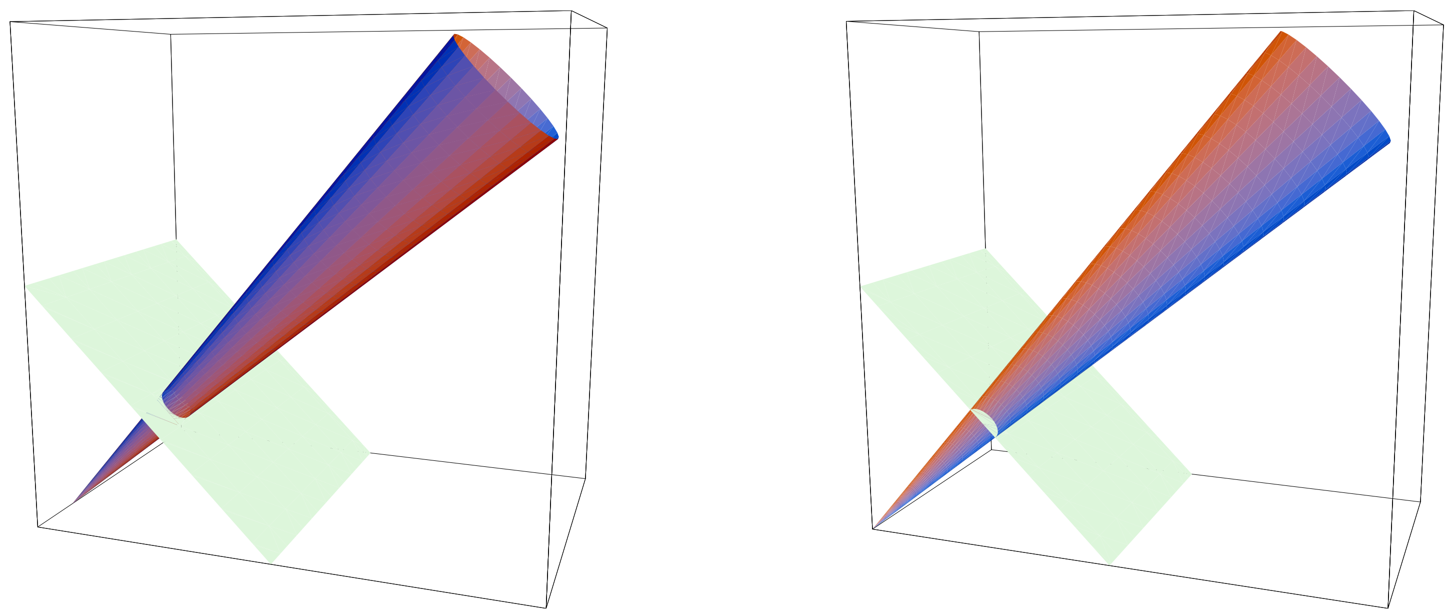
Other quantities can be defined in terms of $L_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}})$.

Here is one:

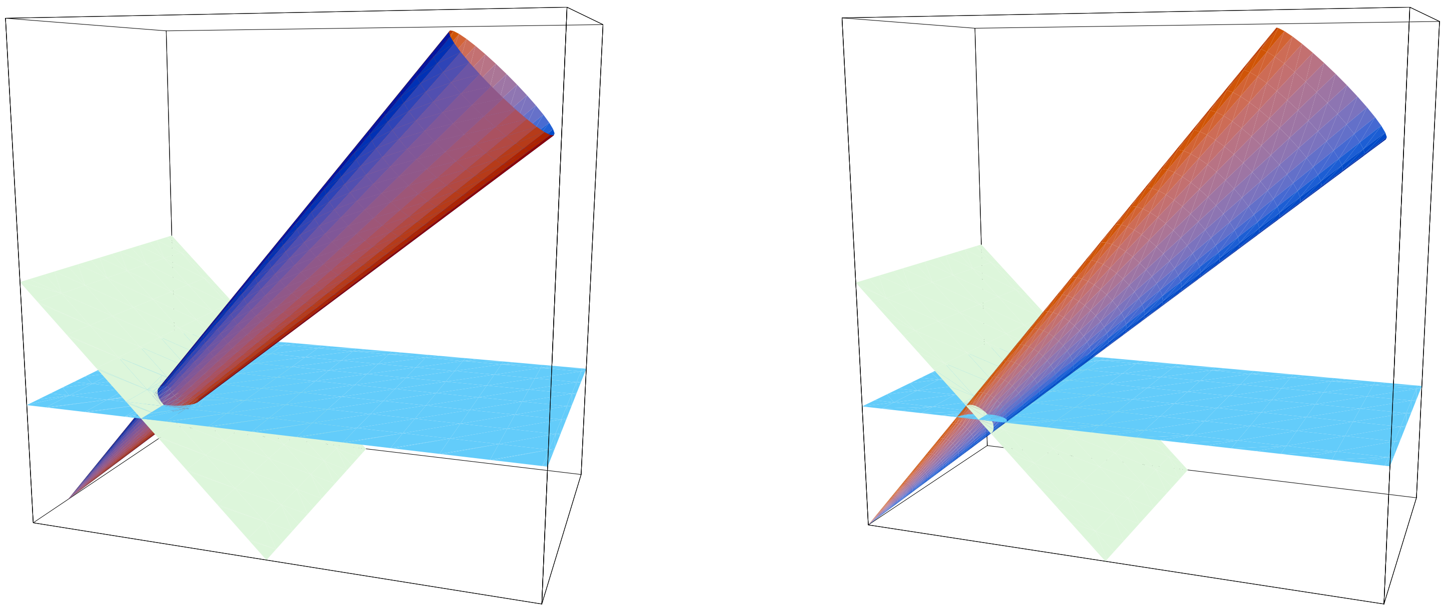
radiance:

$$L(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}}) = \int_0^\infty L_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}}) d\nu$$

This merely integrates over all frequencies, or colors.



Two more figures to explain the definition of **radiance** L and L_ν.



Now the explantion how **irradiance** F and F_ν are related to L and L_ν. Note there is *less* watts per square meter on the blue surface than on the green surface. The contribution of L_ν to F_ν is proportional to the the cosine of the angle between the normals.

Two more:

spectral irradiance:

$$F_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) = \int\limits_{\text{hemisphere}} L_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}})\ \widehat{\mathbf{n}} \cdot \widehat{\mathbf{s}}\ d\Omega$$

irradiance (flux density):

$$F(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) = \int\limits_0^\infty F_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) d\nu = \int\limits_{\text{hemisphere}} L(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}}) \widehat{\mathbf{n}} \cdot \widehat{\mathbf{s}}\ d\Omega$$

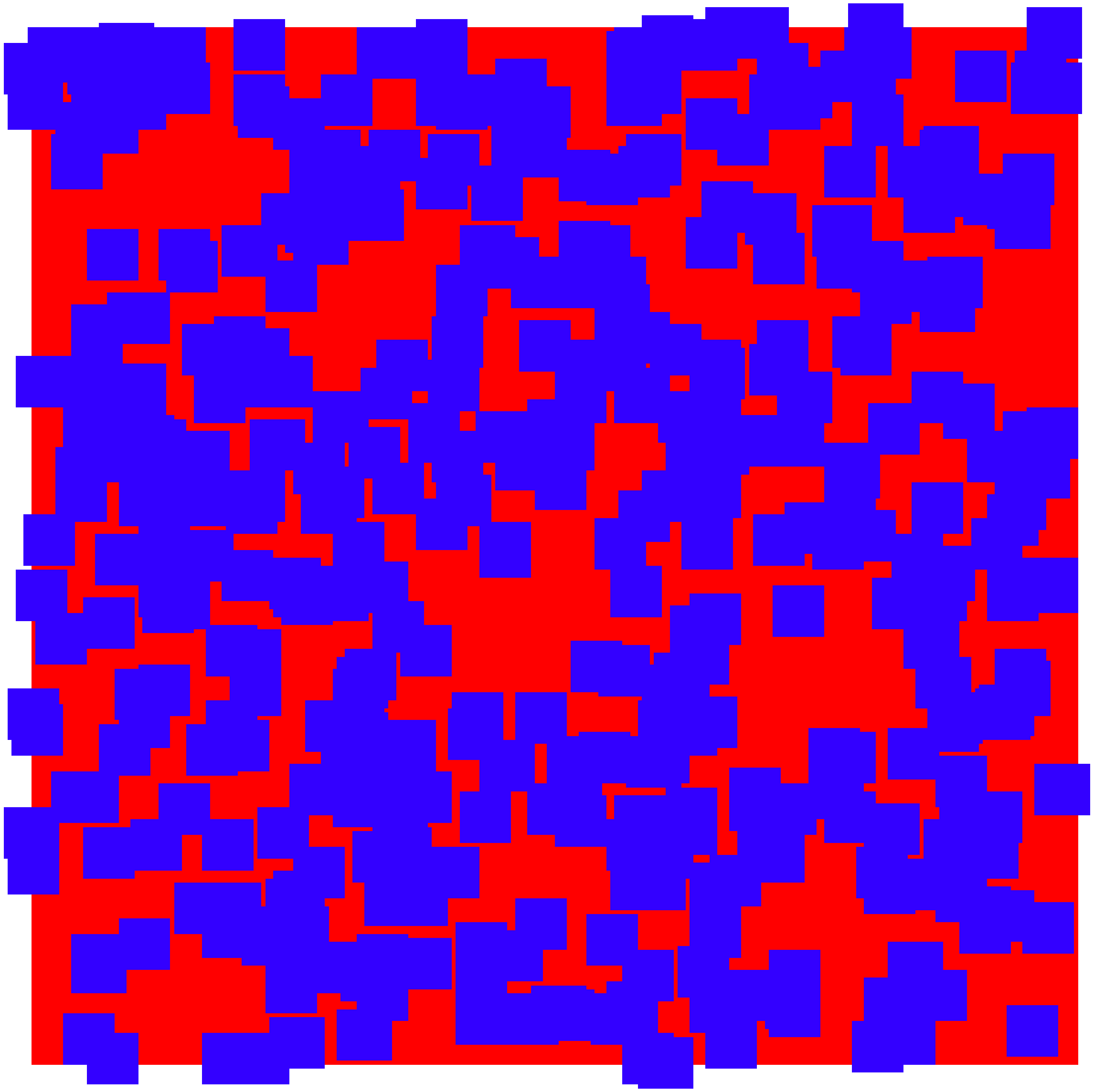
From a perfect blackbody surface:

$$L_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{s}}) = B_\nu(T)$$

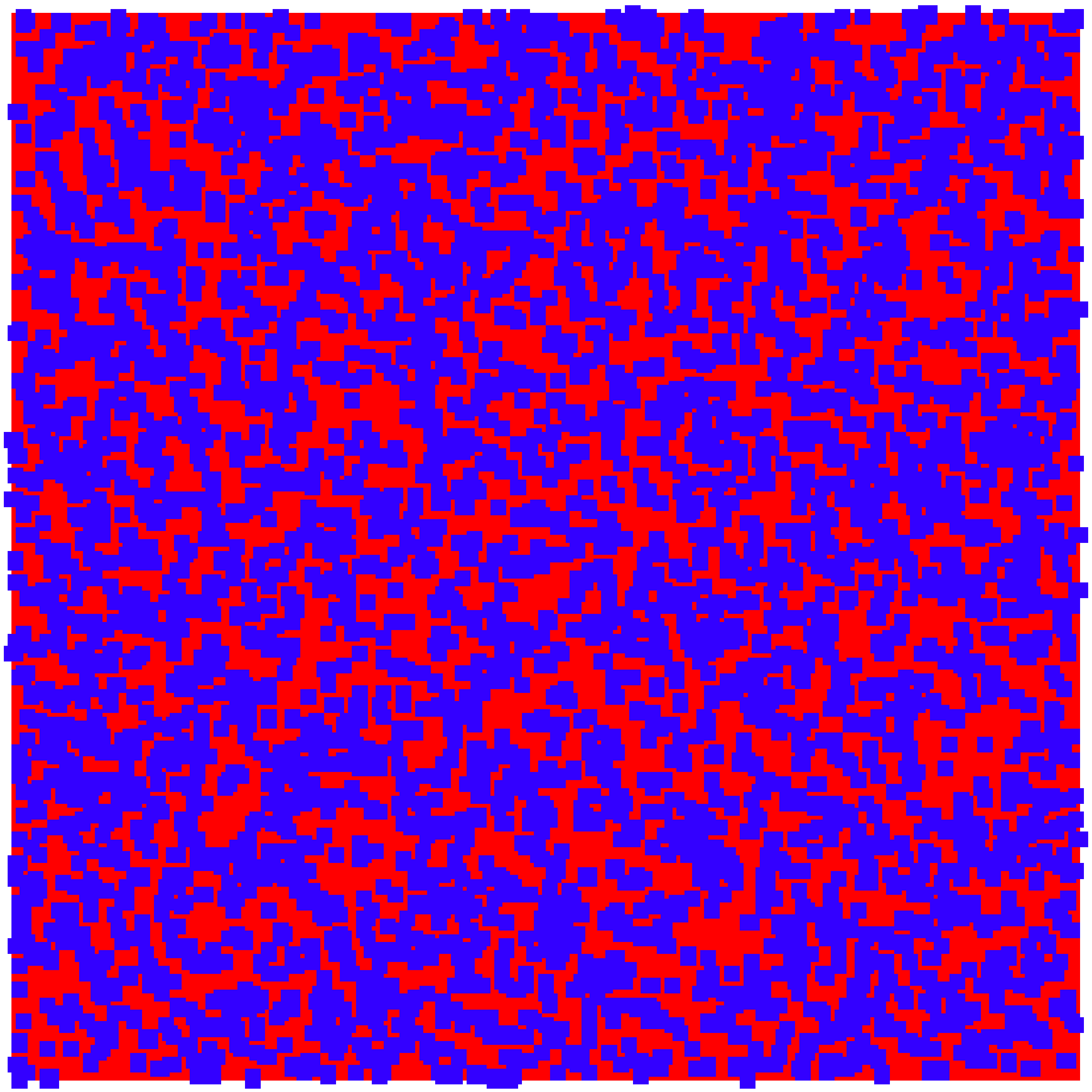
$$F_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) = \int_{\text{hemisphere}} B_\nu(T) \cos\theta \; d\Omega$$

$$F_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) = B_\nu(T) \int_{\theta=0}^{\theta=\pi/2} \cos\theta \; 2\pi \sin\theta \; d\theta$$

$$F_\nu(\vec{\boldsymbol{r}}, \widehat{\mathbf{n}}) = \pi B_\nu(T) \int_{\theta=0}^{\theta=\pi/2} \sin(2\theta) \; d\theta = \pi B_\nu(T)$$

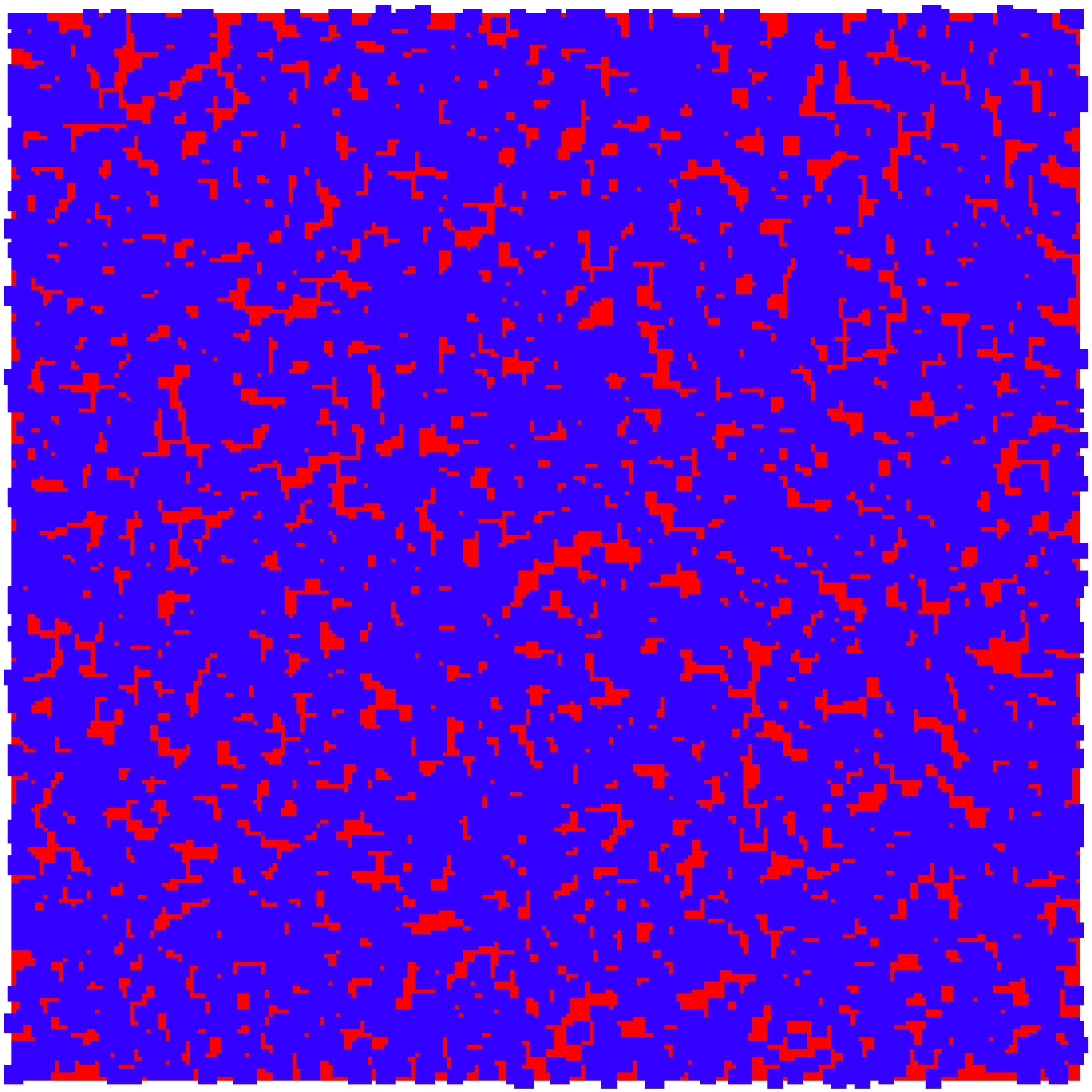


Identical size blue square is cut into 400 pieces of confetti and dropped on the red square. Fraction still red is $\left(1 - \frac{1}{400}\right)^{400} = 0.367419$



... cut into 4000 pieces of confetti and dropped on the red square. Fraction still red is $\left(1 - \frac{1}{4000}\right)^{4000} = 0.367833$

$$\frac{1}{e} = 0.367879$$



... 2 blue pieces cut into 8000 pieces of confetti and dropped on the red square. Fraction still red is $\approx e^{-2} = .135335$

Lambert's Law

$$dL_\nu = -L_\nu \; k_\nu \; \rho_a \; ds$$

ds is the increment in distance along the beam. units: m

ρ_a is the mass density of the radiatively active gas. units: kg m^{-3}

k_ν is the extinction coefficient: units: m^2 kg^{-1}

define *optical path* χ_ν:

$$\chi_\nu(s) = \int_{s_0}^{s} k_\nu(s')\rho_a(s')ds'$$

χ_ν is dimensionless

Lambert's law can be written:

$$\frac{dL_\nu}{d\chi_\nu} = -L_\nu$$

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu}$$

The above solution is sometimes referred to as *Beer's Law*.

Schwarzschild equation recognizes Kirchoff's law and LTE:

$$\frac{dL_\nu}{ds} = -k_\nu \rho_a L_\nu + k_\nu \rho_a B_\nu$$

or

$$\frac{dL_\nu}{d\chi_\nu} = -L_\nu + B_\nu$$

For a constant temperature gas, or B_ν constant:

$$L_\nu(\chi_\nu) = B_\nu + Ae^{-\chi_\nu}$$

$$L_\nu(\chi_\nu) = B_\nu + (L_\nu(0) - B_\nu)e^{-\chi_\nu}$$

$$L_\nu(\chi_\nu) = B_\nu \left(1 - e^{-\chi_\nu}\right) + L_\nu(0)e^{-\chi_\nu}$$

Consider Schwarzschild equation with non-constant B_ν :

$$\frac{dL_\nu}{d\chi_\nu} + L_\nu = B_\nu$$

$$\left(\frac{dL_\nu}{d\chi_\nu} + L_\nu\right) e^{\chi_\nu} = B_\nu e^{\chi_\nu}$$

$$\frac{d}{d\chi_\nu}\left(L_\nu e^{\chi_\nu}\right) = B_\nu e^{\chi_\nu}$$

$$L_\nu(\chi_\nu) e^{\chi_\nu} - L_\nu(0) = \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu) e^{\chi'_\nu} d\chi'_\nu$$

$$L_\nu(\chi_\nu) = L_\nu(0) e^{-\chi_\nu} + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu) e^{-(\chi_\nu-\chi'_\nu)} d\chi'_\nu$$