

## Solution to the Schwarzschild equation for IR

1:

Lecture for Spring 2009  
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Consider an atmosphere of *constant temperature*  $T_a$  and a black-body surface with temperature  $T_s$ . The solution for the upward *spectral radiance* is:

$$L_\nu(\chi_\nu) = B_\nu(T_a) (1 - e^{-\chi_\nu}) + B_\nu(T_s)e^{-\chi_\nu}$$

where the *optical path*  $\chi_\nu$  is:

2:

$$\chi_\nu(s) = \int_{s_0}^s k_\nu(s')\rho_a(s')ds'$$

Similarly,

$$\nu L_\nu(\chi_\nu) = \nu B_\nu(T_a) (1 - e^{-\chi_\nu}) + \nu B_\nu(T_s)e^{-\chi_\nu}$$

In a laboratory, with constant pressure along a tube,  $k_\nu$  could be independent of position. We could write

$$\chi_\nu(s) = k_\nu \int_{s_0}^s \rho_a(s')ds' \equiv k_\nu u$$

3:

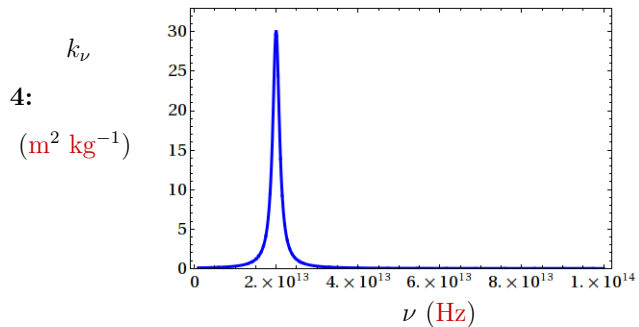
where  $u$  is the *mass path*. A Lorentz profile for  $k_\nu u$  could be written as:

$$k_\nu = k_{\max} \frac{\gamma^2}{\gamma^2 + (\nu - \nu_0)^2}$$

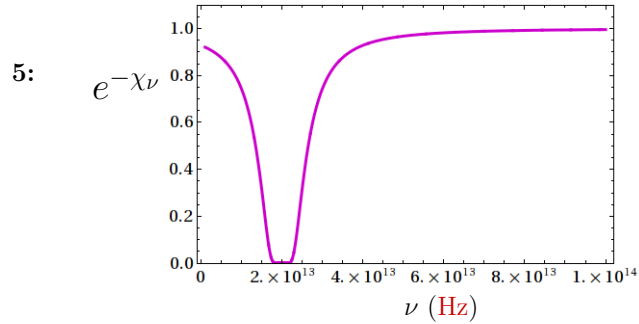
Example  $k_\nu$  with

$$k_{\max} = 30 \text{ m}^2 \text{ kg}^{-1}, \gamma = 0.1 \times 10^{13} \text{ Hz} \text{ and } \nu_0 = 2 \times 10^{13} \text{ Hz}.$$

4:

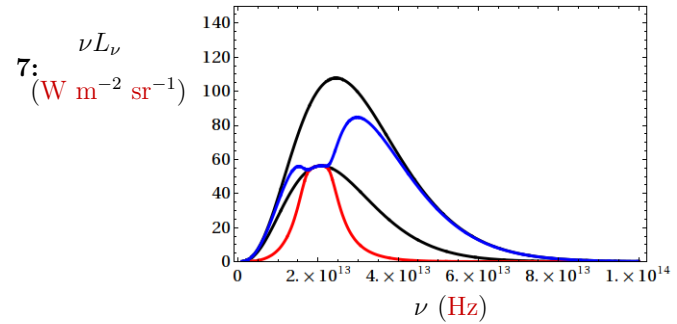


Continue the example, take  $u = 1.0 \text{ kg m}^{-2}$ . With  $\chi_\nu = k_\nu u$ , the maximum in  $\chi_\nu$  is 30. Here the transmittance is  $e^{-\chi_\nu} = e^{-k_\nu u}$ .



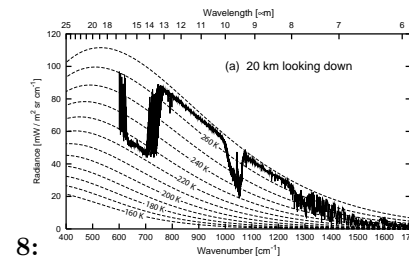
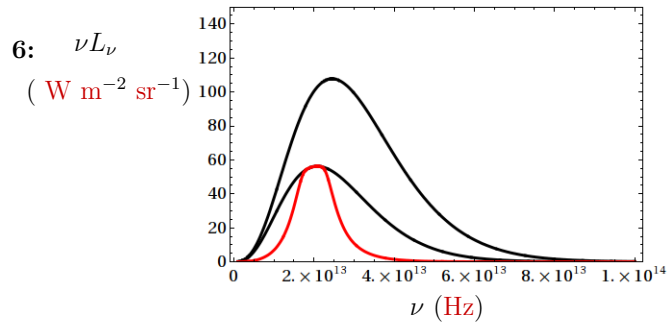
Blue is spectral radiance out of the top of atmosphere,

$$(1 - e^{-\chi_\nu}) \nu B_\nu(T_a) + e^{-\chi_\nu} \nu B_\nu(T_s)$$

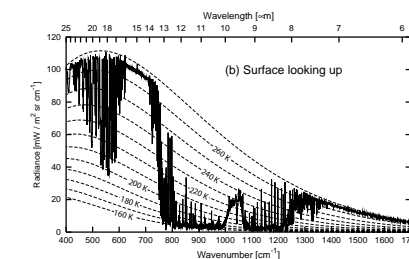


Continue the example with  $T_s = 300 \text{ K}$  and  $T_a = 255 \text{ K}$ . Red is spectral radiance down to the surface, or

$$(1 - e^{-\chi_\nu}) \nu B_\nu(T_a)$$



Looking down through cloud-free atmosphere at polar ice sheet.



Looking up at cloud-free atmosphere from polar ice sheet.

Now consider the solution to the Schwarzschild equation with **non-constant  $T$  and  $B_\nu$**  :

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)} d\chi'_\nu$$

- 9: Consider solution at  $\chi_{\nu m}$ , maximum  $\chi_\nu$  for traversal of the whole atmosphere. Also seek solution “per  $\ln \nu$ ” rather than “per  $\nu$ ”

$$\nu L_\nu(\chi_{\nu m}) = \nu L_\nu(0)e^{-\chi_{\nu m}} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu m}} \nu B_\nu(\chi'_\nu)e^{-(\chi_{\nu m}-\chi'_\nu)} d\chi'_\nu$$

For upward travel:

$$\chi'_\nu = \chi_{\nu m}(1-p) \quad d\chi'_\nu = -\chi_{\nu m} dp \quad \chi_{\nu m} - \chi'_\nu = \chi_{\nu m} p$$

$$11: \quad \nu L_\nu(\chi_{\nu m}) = \nu L_\nu(0)e^{-\chi_{\nu m}} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu m}} \nu B_\nu(\chi'_\nu)e^{-(\chi_{\nu m}-\chi'_\nu)} d\chi'_\nu$$

$$\nu L_\nu(\chi_{\nu m}) = \nu B_\nu(T(1))e^{-\chi_{\nu m}} + \int_{p=1}^{p=0} \nu B_\nu(T(p))e^{-\chi_{\nu m} p} dp$$

For downward travel:

$$\chi'_\nu = \chi_{\nu m} p \quad d\chi'_\nu = \chi_{\nu m} dp \quad \chi_{\nu m} - \chi'_\nu = \chi_{\nu m}(1-p)$$

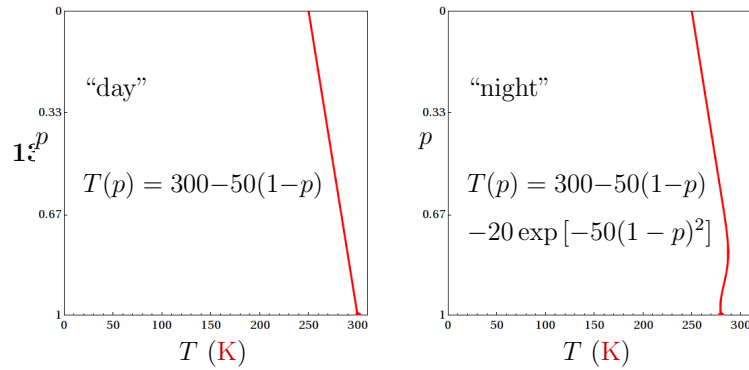
$$12: \quad \nu L_\nu(\chi_{\nu m}) = \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu m}} \nu B_\nu(\chi'_\nu)e^{-(\chi_{\nu m}-\chi'_\nu)} d\chi'_\nu$$

$$\nu L_\nu(\chi_{\nu m}) = \chi_{\nu m} \int_{p=0}^{p=1} \nu B_\nu(T(p))e^{-\chi_{\nu m}(1-p)} dp$$

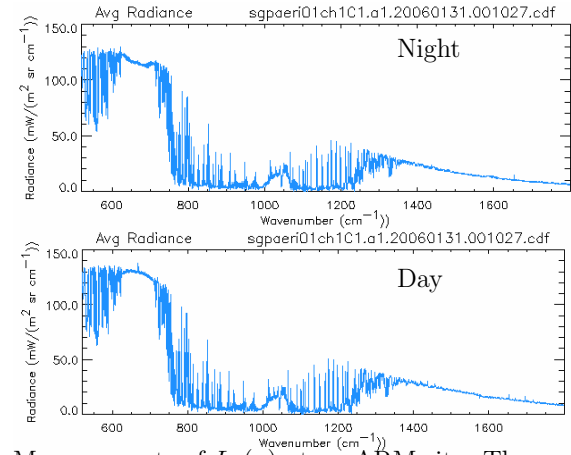
- Let  $p$  be normalized pressure, varying from 1 at the surface to 0 at the top of the atmosphere.

- 10:
- Assume  $T(p)$  is known.
  - For upward travel:  $\chi'_\nu = \chi_{\nu m}(1-p)$
  - For downward travel:  $\chi'_\nu = \chi_{\nu m} p$

Consider some simple profiles for  $T(p)$ :

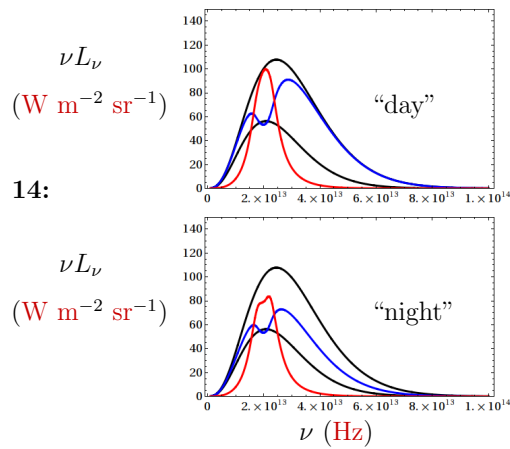


15:



Measurements of  $L_\nu(\nu)$  at an ARM site. The cool boundary layer reduces the spectral radiance in the center of the optically thick part of the CO<sub>2</sub> band.

Solutions for  $L_\nu(\nu)$ :



16:

Recap: the Schwarzschild equation is:

$$\frac{dL_\nu}{d\chi_\nu} = -L_\nu + B_\nu$$

The solution is:

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)} d\chi'_\nu$$

with “optical path” defined:

$$\chi_\nu(s) = \int_{s_0}^s k_\nu(s')\rho_a(s')ds'$$

**Using the transmittance in the solution**

Let the *transmittance* be:

$$\tau_\nu(\chi'_\nu, \chi_\nu) \equiv e^{-(\chi_\nu - \chi'_\nu)} = e^{\chi'_\nu - \chi_\nu}$$

**17:** The solution can be written:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)\tau(\chi'_\nu, \chi_\nu)d\chi'_\nu$$

But note:

$$\frac{d\tau_\nu}{d\chi'_\nu} = \frac{d}{d\chi'_\nu} e^{\chi'_\nu - \chi_\nu} = e^{\chi'_\nu - \chi_\nu} = \tau_\nu(\chi'_\nu, \chi_\nu)$$

The solution can be written:

**18:**

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \int_{\tau'_\nu=\tau_\nu(0, \chi_\nu)}^{\tau'_\nu=1} B_\nu(T)d\tau'_\nu$$

Check case of constant  $T$ :

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + B_\nu(T)(1 - \tau_\nu(0, \chi_\nu))$$

Contemplate numerical solution:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \sum_{n=1}^N B_\nu(T_n) [\tau_\nu(\chi_{\nu, n+}, \chi_\nu) - \tau_\nu(\chi_{\nu, n-}, \chi_\nu)]$$

**19:**

Where  $T_n$  is the temperature of layer  $n$ ,  $\chi_{\nu, n+}$  is the optical path for the side of layer nearest the detector, and  $\chi_{\nu, n-}$  is the optical path for the side of layer farthest from the detector.

With monochromatic transmittance defined as

$$\tau_\nu(\chi'_\nu, \chi_\nu) \equiv e^{-(\chi_\nu - \chi'_\nu)} = e^{\chi'_\nu - \chi_\nu}$$

We demonstrate the important principle about the product of transmittances:

**20:**

$$\tau_\nu(a, b) = e^{a-b} \quad \tau_\nu(b, c) = e^{b-c}$$

$$\tau_\nu(a, b)\tau_\nu(b, c) = e^{a-b}e^{b-c} = e^{a-c} = \tau_\nu(a, c)$$

Compare with mass path:

$$u(a, b) + u(b, c) = u(a, c)$$

Use the transmittance property:

$$\tau_\nu(\chi_{\nu,n-}, \chi_\nu) = \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})\tau_\nu(\chi_{\nu,n+}, \chi_\nu)$$

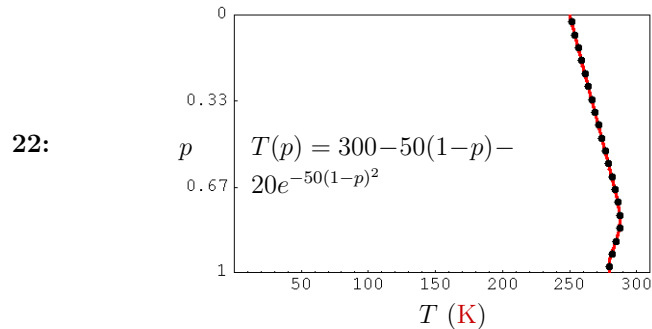
21:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \sum_{n=1}^N B_\nu(T_n) [\tau_\nu(\chi_{\nu,n+}, \chi_\nu) - \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})\tau_\nu(\chi_{\nu,n+}, \chi_\nu)]$$

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \sum_{n=1}^N B_\nu(T_n) [1 - \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})] \tau_\nu(\chi_{\nu,n+}, \chi_\nu)$$

The term in [ ] is the *emittance* of the n'th layer.

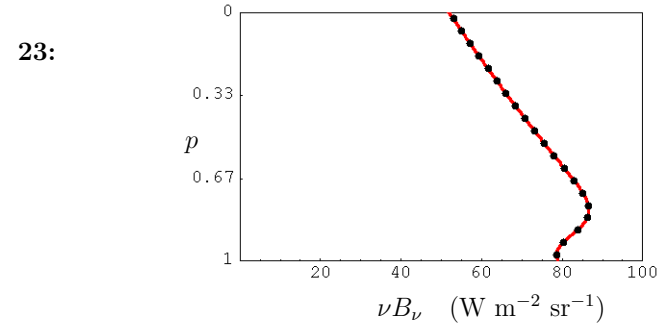
Consider a very simple profile for  $T(p)$  with a nocturnal boundary layer:



Will model with N=20 discrete isothermal layers.

We take  $\nu = 2 \times 10^{13}$  Hz solely for the purpose of providing typical values for  $B_\nu(T)$ . But otherwise  $\nu$  is variable:

we allow  $\nu$  to vary so that we can study  $\chi_{\nu m}$  large or small.



The optical path through one of the  $N$  layers is:

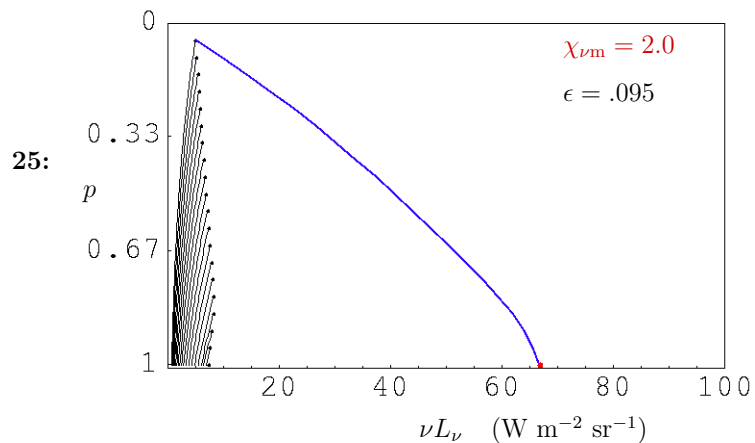
$$\Delta\chi_\nu = \frac{\chi_{\nu m}}{N}$$

24: The spectral radiance emerging from the n'th layer is:

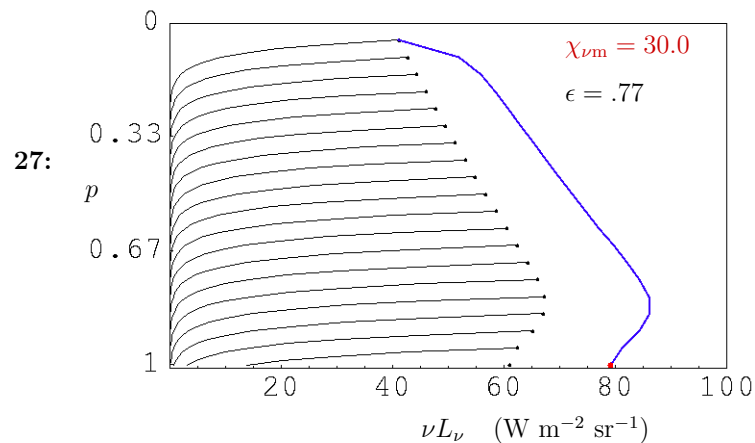
$$\nu L_\nu = \nu B_\nu(T_n)(1 - e^{-\Delta\chi_\nu}) \equiv \nu B_\nu(T_n)\epsilon$$

This spectral radiance will exponentially decay as it travels toward the ground.

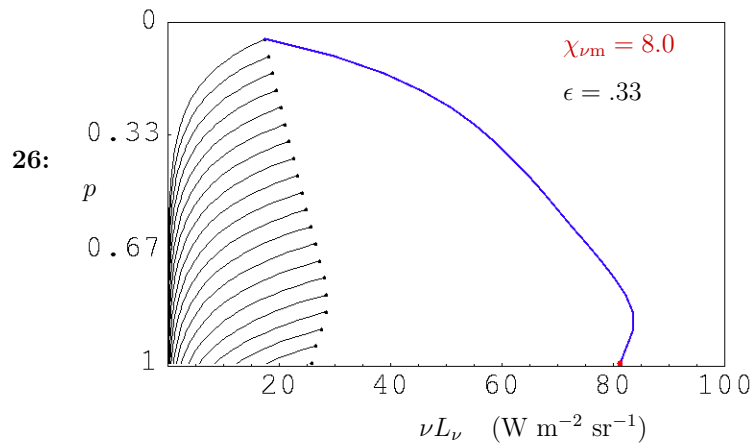
Black:  $\nu L_\nu$  emerging from the isothermal layers  
Blue: total  $\nu L_\nu$ . Red: total  $\nu L_\nu$  at ground.



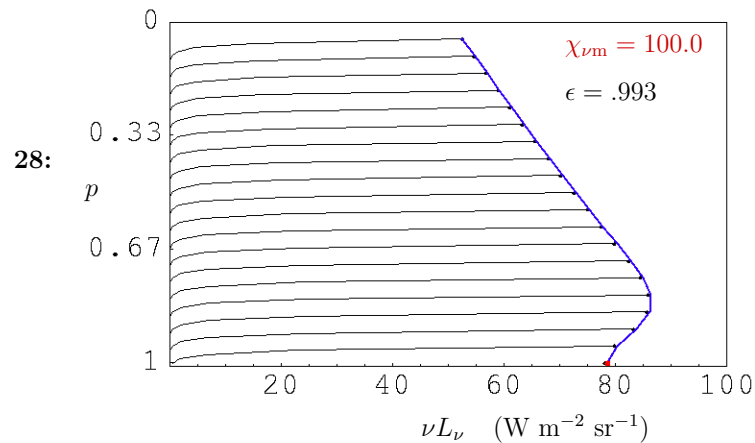
Black:  $\nu L_\nu$  emerging from the isothermal layers  
Blue: total  $\nu L_\nu$ . Red: total  $\nu L_\nu$  at ground.



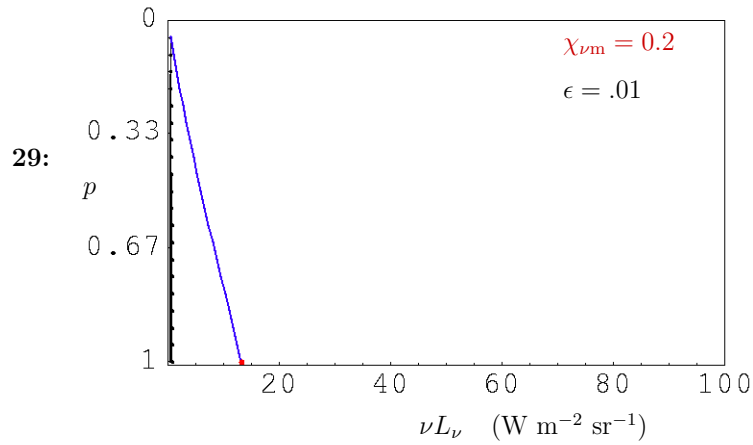
Black:  $\nu L_\nu$  emerging from the isothermal layers  
Blue: total  $\nu L_\nu$ . Red: total  $\nu L_\nu$  at ground.



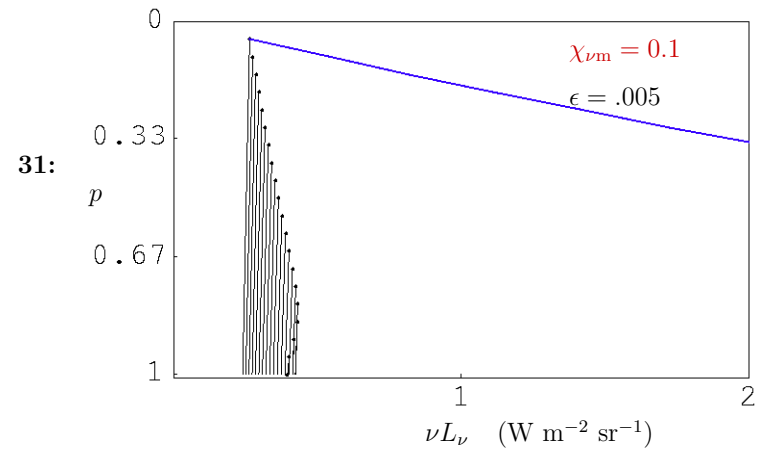
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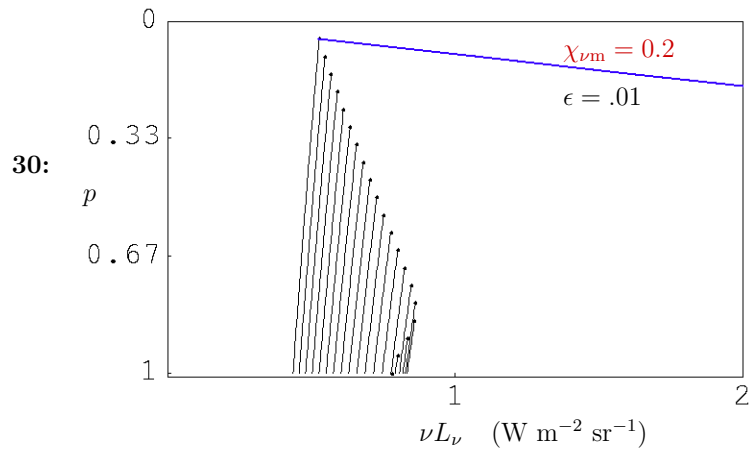
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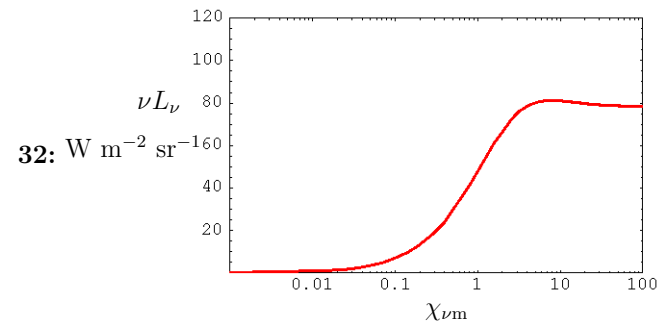
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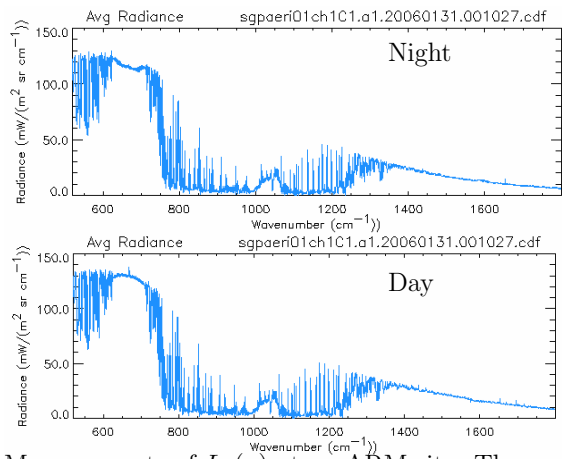


$\nu L_\nu(\chi_{\nu m})$  into the ground from the discrete 20-layer model:



This plot is nearly indistinguishable from that of the complete model.

33:



Measurements of  $L_\nu(\nu)$  at an ARM site. The cool boundary layer reduces the spectral radiance in the center of the optically thick part of the  $\text{CO}_2$  band.